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All bicovariant differential calculuses on $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$

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Abstract. All bicovariant first-order differential calculuses on the quantum group $GL_q(3,\mathbb{C})$ are determined. There are two distinct one-parameter families of calculuses. In terms of a suitable basis of 1-forms the commutation relations can be expressed with the help of the R-matrix of $GL_q(3,\mathbb{C})$. Some calculuses induce bicovariant differential calculuses on $SL_q(3,\mathbb{C})$ and on real forms of $GL_q(3,\mathbb{C})$. For the generic deformation parameter q there are six calculuses on $SL_q(3,\mathbb{C})$, on $SU_q(3)$ there are only two. The classical limit $q\to 1$ of bicovariant calculuses on $SL_q(3,\mathbb{C})$ is not the ordinary calculus on $SL(3,\mathbb{C})$. One obtains a deformation of it which involves the Cartan–Killing metric.

1. Introduction

In recent years 'non-commutative geometry' (see [1, 2] for some aspects of it) appeared as a new branch of geometry and a new framework for physical model building. It has its origin in the basic observation that a manifold (respectively, a topological space) is completely characterized by the algebra of functions on it, viewed as an abstract commutative (C^*-) algebra. Geometrical concepts can be understood as algebraic structures on this algebra and then generalized to non-commutative algebras (for which there is no longer an underlying topological space).

In differential geometry an important role is played by Lie groups which correspond to commutative Hopf algebras [3, 4]. 'Quantum groups' are non-commutative Hopf algebras. Examples are obtained as deformations of classical groups (as Hopf algebras) [5–8]. In particular, they provide us with new symmetry concepts which are of relevance, in particular, in the context of conformal field theories and quantum integrable models.

The differential geometry of Lie groups (and their co-set spaces) enters the mathematical modelling of physical theories. In particular, this is the case for classical gauge theories formulated in terms of connections on principal fibre bundles, and for Kaluza–Klein theories. First steps have been made to generalize the corresponding notions to the realm of non-commutative geometry (see [9–11], for example). There is some hope of obtaining interesting 'deformations' of physical models in this way, in particular for elementary particle physics and gravitation.

A central part of such a programme is to develop a differential calculus on quantum groups. This has been done by Woronowicz [12]. He introduced the notion of bicovariance as a natural condition to reduce the number of possible differential algebras associated with a given quantum group. In the meantime a large number of papers appeared dealing with

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examples of bicovariant differential calculuses on special (classes of) quantum groups (see [13] for an extensive list of references). However, one would like to have a complete description of all possible bicovariant differential calculuses on certain quantum groups rather than just a collection of examples. For the two-parameter quantum group $GL_{p,q}(2,\mathbb{C})$ and related subgroups this was achieved in [13, 14]. We used similar methods to determine all bicovariant (first-order) differential calculuses on $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$†. Examples of bicovariant differential calculuses on $GL_q(3,\mathbb{C})$ have already been presented in [15].

The classical limit $q \to 1$ leads to a Hopf algebraic description of the Lie groups $GL(3,\mathbb{C})$ and $SL(3,\mathbb{C})$. One might expect the usual differential geometry of these groups to be recovered in this limit. However, for $q \to 1$ we obtain an interesting deformation of the ordinary differential calculus on $SL(3,\mathbb{C})$ (see also [13] for the case of $SL(2,\mathbb{C})$). Functions on the group no longer commute with 1-forms, the commutation relations involve the Cartan–Killing metric. This observation may be taken as a starting point for further investigations aiming at the notion of a 'quantum group metric'.

Section 2 recalls the notions of differential calculus and bicovariance on quantum groups. In section 3 we briefly review the Hopf algebraic structure of $GL_q(3,\mathbb{C})$. The central part of our work is section 4 which deals with the determination of all bicovariant differential calculuses on $GL_q(3,\mathbb{C})$ and a discussion of some of their properties. In section 5 we turn to the investigation of bicovariant calculuses on quantum subgroups of $GL_q(3,\mathbb{C})$. Section 6 is devoted to the classical limit of bicovariant differential calculuses on $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$. Finally, in section 7 we relate our results to the work of other authors and try to give a perspective for further studies.

2. Differential calculus on quantum groups

We first recall the definition of a (first-order) differential calculus on an associative algebra $\mathcal{A}$ and specify later to the case of a Hopf algebra (respectively, a quantum group) [12].

Definition. Let $\mathcal{A}$ be an associative unital algebra. An $\mathcal{A}$-bimodule Γ together with a linear map $\mathrm{d}: \mathcal{A} \longrightarrow \Gamma$ is called first-order differential calculus over $\mathcal{A}$ iff

(i) $\mathrm{d}(ab) = (\mathrm{d}a)b + a(\mathrm{d}b)$ for all $a, b \in \mathcal{A}$,
(ii) $\mathrm{d}\mathcal{A}$ generates Γ as a left $\mathcal{A}$-module.

Two first-order differential calculuses (Γ, d) and $(\tilde{\Gamma}, \tilde{\mathrm{d}})$ over $\mathcal{A}$ are said to be equivalent iff there exists a bimodule isomorphism $\zeta : \Gamma \longrightarrow \tilde{\Gamma}$ with $\tilde{\mathrm{d}} = \zeta \circ \mathrm{d}$. This definition generalizes the classical notion of first-order differential forms. We will therefore call the elements of Γ 1-forms.

Let us now turn to the case of a Hopf algebra. Besides the multiplication and the unit element a quantum group carries the following additional structure:

$$\begin{array}{ll} \Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A} & \text{(co-product)} \\ \varepsilon : \mathcal{A} \longrightarrow \mathbb{C} & \text{(co-unit)} \\ S : \mathcal{A} \longrightarrow \mathcal{A} & \text{(antipode)}. \end{array} \tag{2.1}$$

The first two maps are algebra homomorphisms, the third is an algebra antihomomorphism. These maps have to fulfil certain axioms which we need not recall here (cf [3, 4, 7]). In the commutative case they encode the group structure of the underlying group manifold in the algebraic structure of the algebra of functions on the group. In particular, the co-product

† These results were communicated at the spring meeting of the Deutsche Physikalische Gesellschaft in Hamburg, March 1994.

translates the group multiplication and can be used to reformulate the left and right action of the group on itself. One may now ask whether there are corresponding generalizations of the induced actions of the group on differential forms. This leads to the notion of bicovariance which is briefly recalled in the sequel.

Definition. Let $\mathcal{A}$ be a Hopf algebra with unit element $\mathbf{1}$. A first-order differential calculus $(\Gamma,\ \mathrm{d})$ over $\mathcal{A}$ is called bicovariant iff there are linear maps $\Delta_{\mathcal{L}}:\Gamma\to\mathcal{A}\otimes\Gamma$ and $\Delta_{\mathcal{R}}:\Gamma\to\Gamma\otimes\mathcal{A}$, which are called left and right coactions, such that

$$\Delta_{\mathcal{L}}(a\,\mathrm{d}b)=\Delta(a)(\mathrm{id}\otimes\mathrm{d})\Delta(b) \tag{2.2}$$
$$\Delta_{\mathcal{R}}(a\,\mathrm{d}b)=\Delta(a)(\mathrm{d}\otimes\mathrm{id})\Delta(b)\,. \tag{2.3}$$

An element $\omega\in\Gamma$ is said to be left-/right-invariant iff

$$\Delta_{\mathcal{L}}(\omega)=\mathbf{1}\otimes\omega \tag{2.4}$$
$$\Delta_{\mathcal{R}}(\omega)=\omega\otimes\mathbf{1} \tag{2.5}$$

respectively. ω is called bi-invariant iff (2.4) and (2.5) hold simultaneously.

A bicovariant differential calculus is a special case of a structure called bicovariant bimodule, which is by definition an $\mathcal{A}$ -bimodule Γ together with linear maps $\Delta_{\mathcal{L}}:\Gamma\to\mathcal{A}\otimes\Gamma$ and $\Delta_{\mathcal{R}}:\Gamma\to\Gamma\otimes\mathcal{A}$ satisfying

$$\begin{aligned}
&\Delta_{\mathcal{L}}(a\varrho b)=\Delta(a)\Delta_{\mathcal{L}}(\varrho)\Delta(b) && \Delta_{\mathcal{R}}(a\varrho b)=\Delta(a)\Delta_{\mathcal{R}}(\varrho)\Delta(b)\\
&(\mathrm{id}\otimes\Delta_{\mathcal{L}})\circ\Delta_{\mathcal{L}}=(\Delta\otimes\mathrm{id})\circ\Delta_{\mathcal{L}} && (\Delta_{\mathcal{R}}\otimes\mathrm{id})\circ\Delta_{\mathcal{R}}=(\mathrm{id}\otimes\Delta)\circ\Delta_{\mathcal{R}}\\
&(\varepsilon\otimes\mathrm{id})\circ\Delta_{\mathcal{L}}(\varrho)=\varrho && (\mathrm{id}\otimes\varepsilon)\circ\Delta_{\mathcal{R}}(\varrho)=\varrho
\end{aligned}$$

and

$$(\mathrm{id}\otimes\Delta_{\mathcal{R}})\circ\Delta_{\mathcal{L}}=(\Delta_{\mathcal{L}}\otimes\mathrm{id})\circ\Delta_{\mathcal{R}}\,.$$

For $\Delta_{\mathcal{L}}$ and $\Delta_{\mathcal{R}}$ given by (2.2) and (2.3) these identities are satisfied. It turns out that the whole structure of a bicovariant bimodule Γ can be conveniently described by its left- (or right-) invariant elements. We introduce the left and right convolution products, defined for $f\in\mathcal{A}'=\mathrm{Hom}(\mathcal{A},\mathbb{C})$ and $a\in\mathcal{A}$ by

$$f*a=(\mathrm{id}\otimes f)\Delta(a) \tag{2.6}$$
$$a*f=(f\otimes\mathrm{id})\Delta(a) \tag{2.7}$$

and recall some results from [12].

Proposition 2.1. Let (Γ,d) be a bicovariant bimodule over the Hopf algebra $\mathcal{A}$. The set of all left-invariant elements of Γ, called ${}_{\mathrm{inv}}\Gamma$, is a linear subspace of Γ. Let $\{\omega^I\}_{I\in\mathcal{I}}$ be a basis of ${}_{\mathrm{inv}}\Gamma$. Then:

(i) Any $\rho\in\Gamma$ can uniquely be written as $\rho=a_I\,\omega^I$ with $a_I\in\mathcal{A}$.
(ii) There exist linear functionals ${f^I}_J\in\mathcal{A}'$ such that

$$\omega^I a=({f^I}_J*a)\omega^J \qquad \forall I\in\mathcal{I} \qquad \forall a\in\mathcal{A}\,. \tag{2.8}$$

The functionals are uniquely determined by (2.8) and fulfil the relations

$${f^I}_J(ab)={f^I}_K(a){f^K}_J(b) \tag{2.9}$$
$${f^I}_J(\mathbf{1})={\delta^I}_J\,. \tag{2.10}$$

(iii) The right coaction on the basis $\{\omega^I\}_{I\in\mathcal{I}}$ is given by

$$\Delta_{\mathcal{R}}(\omega^I) = \omega^J \otimes M_J{}^I \tag{2.11}$$

with $M_J{}^I \in \mathcal{A}$ satisfying

$$\Delta(M_I{}^J) = M_I{}^K \otimes M_K{}^J \tag{2.12}$$

$$\varepsilon(M_I{}^J) = \delta_I{}^J . \tag{2.13}$$

(iv) Bicovariance implies

$$M_I{}^J(a * f^I{}_K) = (f^J{}_I * a)M_K{}^I \qquad \forall a \in \mathcal{A} \qquad \forall J, K \in \mathcal{I}. \tag{2.14}$$

In this short exposition we will not consider the higher-order differential calculus. We only mention that every bicovariant first-order differential calculus admits an extension to a differential algebra containing forms of arbitrary order (cf [12, 16]).

3. The quantum group $GL_q(3,\mathbb{C})$

Deformations of Lie groups can be obtained by introducing a non-commutative multiplication structure on the related Hopf algebra. This usually involves deformation parameters. Corresponding multi-parameter deformations of (the algebra of functions on) the general linear groups are known (cf [17–19]). Examples of differential calculuses have been constructed on some of them [19–21]. Here we concentrate on the standard one-parameter deformation of the algebra of functions on $GL(3,\mathbb{C})$ [22]. This is the algebra $\mathcal{A} := \mathrm{Fun}_q(GL(3,\mathbb{C}))$ generated by

(i) nine non-commuting entities $z^i{}_j$, $i, j = 1, 2, 3$, which we arrange as a matrix $Z = (z^i{}_j)$. Their commutation relations are

$$\begin{array}{lll} j<k & & z^i{}_j z^i{}_k = q\, z^i{}_k z^i{}_j \\ i<k & & z^i{}_j z^k{}_j = q\, z^k{}_j z^i{}_j \\ i<k & j>l & z^i{}_j z^k{}_l = z^k{}_l z^i{}_j \\ i<k & j<l & z^i{}_j z^k{}_l = z^k{}_l z^i{}_j + (q-q^{-1})\, z^k{}_j z^i{}_l . \end{array} \tag{3.1}$$

For $q \to 1$ all the matrix elements of Z commute with each other (classical limit). Sometimes it is convenient to treat the indices ${}^i{}_j$ of $z^i{}_j$ as 'composite indices' taking values $1, \ldots, 9$ (via ${}^1{}_1 \to 1$, ${}^1{}_2 \to 2$, ${}^1{}_3 \to 3$, ${}^2{}_1 \to 4$, etc).

(ii) the unit $\mathbf{1}$ and the inverse $\mathcal{D}^{-1}$ of the quantum determinant

$$\mathcal{D} = z^1z^5z^9 + q^2z^2z^6z^7 + q^2z^3z^4z^8 - qz^1z^6z^8 - q^3z^3z^5z^7 - qz^2z^4z^9 \tag{3.2}$$

which is central in $\mathcal{A}$.

This non-commutative algebra can be endowed with a co-product, co-unit and antipode *in the following way:*

$$\begin{array}{lll} \Delta(z^i{}_j) = z^i{}_k \otimes z^k{}_j & \Delta(1) = \mathbf{1}\otimes\mathbf{1} & \Delta(\mathcal{D}^{-1}) = \mathcal{D}^{-1}\otimes\mathcal{D}^{-1} \\ \varepsilon(z^i{}_j) = \delta^i{}_j & \varepsilon(1) = 1 & \varepsilon(\mathcal{D}^{-1}) = 1 \\ S(z^i{}_j) = (S(Z))^i{}_j & S(1) = \mathbf{1} & S(\mathcal{D}^{-1}) = \mathcal{D} \end{array} \tag{3.3}$$

where the summation convention is used and the matrix $S(Z)$ is given by

$$S(Z) = \mathcal{D}^{-1} \begin{pmatrix} z^5z^9 - qz^6z^8 & -q^{-1}z^2z^9 + z^3z^8 & q^{-2}z^2z^6 - q^{-1}z^3z^5 \\ -qz^4z^9 + q^2z^6z^7 & z^1z^9 - qz^3z^7 & -q^{-1}z^1z^6 + z^3z^4 \\ q^2z^4z^8 - q^3z^5z^7 & -qz^1z^8 + q^2z^2z^7 & z^1z^5 - qz^2z^4 \end{pmatrix} . \tag{3.4}$$

$(\mathcal{A}, \cdot, \mathbf{1}, \Delta, \varepsilon, S)$ then constitutes a Hopf algebra which may formally be regarded as an algebra of 'functions' on some (fictitious) space $GL_q(3,\mathbb{C})$.

Remark. In a similar way one obtains the Hopf algebra $\mathrm{Fun}_q(GL(n,\mathbb{C}))$ using n^2 generators $Z = (z^i{}_j)$. Let $Z_1 = Z \otimes I$, $Z_2 = I \otimes Z$ where I is the $n \times n$ unit matrix. The relations (3.1) can be written in compact form

$$R_{12} Z_1 Z_2 = Z_2 Z_1 R_{12} \tag{3.5}$$

with the help of the nonsingular complex matrix $R \in \mathrm{M}(n^2, \mathbb{C})$

$$R = \sum_{i,j=1}^{n} q^{\delta^i{}_j} e_i{}^i \otimes e_j{}^j + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{n} e_i{}^j \otimes e_j{}^i \qquad q \in \mathbb{C} \setminus \{0\} \tag{3.6}$$

where the matrices $e_i{}^j$ are defined by $(e_i{}^j)^k{}_l = \delta^k_i \delta^j{}_l$. This matrix satisfies the quantum Yang–Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} . \tag{3.7}$$

The antipode of $\mathcal{A}$ is invertible. Defining a diagonal matrix $D = \mathrm{diag}(1, q^2, \ldots, q^{2(n-1)})$ one has

$$S^{-1}(Z) = D^{-1} S(Z) D . \tag{3.8}$$

4. Bicovariant differential calculus on $GL_q(3,\mathbb{C})$

Let (Γ, d) be a first-order differential calculus over $\mathcal{A} := \mathrm{Fun}_q(GL(3,\mathbb{C}))$. Γ is generated by the differentials $\mathrm{d}z^i{}_j$ $(i, j = 1, 2, 3)$ as an $\mathcal{A}$-bimodule. The differentials of the other generators are obtained using the Leibniz rule:

$$\mathrm{d}\mathbf{1} = 0 \tag{4.1}$$

$$\mathrm{d}\mathcal{D}^{-1} = -\mathcal{D}^{-1} (\mathrm{d}\mathcal{D}) \mathcal{D}^{-1} . \tag{4.2}$$

To mimic the case of (commutative) differential geometry it is natural to require that Γ is generated by $\mathrm{d}z^i{}_j$ $(i, j = 1, 2, 3)$ as a left $\mathcal{A}$-module. This assumption will be made in the sequel.

Now we proceed along the lines of [14] with emphasis on the fundamental results of [12].

4.1. The left-invariant Maurer–Cartan 1-forms

In order to determine the most general commutation relations of elements of Γ with elements of $\mathcal{A}$ we use a convenient basis of Γ. It consists of the quantum analogues of the Maurer–Cartan 1-forms defined by

$$\theta^i{}_j = S(z^i{}_k) \, \mathrm{d}z^k{}_j . \tag{4.3}$$

The relevant properties of these 1-forms are summarized in

Lemma 4.1. (i) The 1-forms $\theta^i{}_j$ are left-invariant, i.e.

$$\Delta_{\mathcal{L}}(\theta^i{}_j) = \mathbf{1} \otimes \theta^i{}_j . \tag{4.4}$$

(ii) The set $\mathcal{B} := \{\theta^i{}_j | i, j = 1, 2, 3\}$ is a basis of ${}_{\mathrm{inv}}\Gamma$ as a $\mathbb{C}$-vector space.
(iii) For the right coaction on $\theta^i{}_j$, one finds

$$\Delta_{\mathcal{R}}(\theta^i{}_j) = \theta^m{}_n \otimes M_m{}^{ni}{}_j \qquad M_m{}^{ni}{}_j := S(z^i{}_m) z^n{}_j \ \in \mathcal{A} . \tag{4.5}$$

By forming composite indices from the matrix indices (see section 3) one obtains (2.11) with $M_J{}^I$ satisfying (2.12) and (2.13).

Remark. Using (3.8) one can verify the identity $\sum_i q^{-2i} S(z^i{}_l) z^k{}_i = q^{-2k}\delta^k_l$. This shows that $\mathrm{Tr}_q\,\theta = \sum_i q^{6-2i}\theta^i{}_i$ is a bi-invariant element of Γ .

4.2. Structure of the commutation relations

Since the Maurer–Cartan 1-forms $\theta^i{}_j$ form a basis of the space of all left-invariant 1-forms ${}_{\mathrm{inv}}\Gamma$ we have uniquely determined linear functionals $f^I{}_J \in \mathcal{A}'$, $1 \leqslant I, J \leqslant 9$, such that

$$\theta^I a = (f^I{}_J * a)\theta^J = ((\mathrm{id} \otimes f^I{}_J) \circ \Delta(a))\theta^J \tag{4.6}$$

for all $a \in \mathcal{A}$ (proposition 2.1). Because of (2.9) and (2.10) these functionals provide us with a representation $\mathcal{F} : \mathcal{A} \longrightarrow \mathrm{M}(9, \mathbb{C})$. The 'fundamental matrices'

$$\mathcal{F}(z^i{}_j) = (f^I{}_J(z^i{}_j))_{I,J=1,\ldots,9} \tag{4.7}$$

completely and uniquely specify the first-order differential calculus (using the equivalence definition of section 2.1).

There are restrictive conditions which a set of matrices has to satisfy in order to be the fundamental matrices of a bicovariant differential calculus on $\mathcal{A}$:

(i) Consistency with the commutation relations of $\mathcal{A}$:
By differentiating the commutation relations (3.5) one obtains

$$0 = \mathrm{d}(RZ_1Z_2 - Z_2Z_1R) = R\mathrm{d}Z_1Z_2 + RZ_1\mathrm{d}Z_2 - \mathrm{d}Z_2Z_1R - Z_2\mathrm{d}Z_1R\,.$$

After convertion of the differentials into Maurer–Cartan forms and commuting all algebra elements to the left we get conditions for the values $f^I{}_J(z^i{}_j)$ of the functionals $f^I{}_J$.

(ii) Bicovariance conditions (2.14):
Inserting the algebra generators $z^i{}_j$ in (2.14) and using (4.6) further conditions are obtained for the values $f^I{}_J(z^i{}_j)$.

(iii) Representation properties of the functionals $f^I{}_J$:
Acting with $\mathcal{F}$ on the commutation relations (3.5) and using the representation property of $\mathcal{F}$ leads to further equations for the matrices $\mathcal{F}(z^i{}_j)$. These are nonlinear equations, in general. Furthermore, $\mathcal{F}(\mathcal{D})$ has to be invertible in $\mathrm{M}(9, \mathbb{C})$.

Using the conditions (i)–(iii) one can derive the most general set of matrices $\mathcal{F}(z^i{}_j)$ which determines a bicovariant differential calculus. For this purpose we used the computer algebra software *Reduce*. It is convenient to solve the equations resulting from (i) and (ii) first because they are linear in the matrix elements. Using finally the equations resulting from condition (iii) we are led to the following results.

4.3. Results

Proposition 4.2. Let $q \in \mathbb{C} \setminus \{0, \pm 1, \pm \mathrm{i}\}$. Then all bicovariant differential calculuses on $GL_q(3, \mathbb{C})$ are contained in two disjoint one-parameter families of calculuses denoted by $\Gamma_\nu(t)$, $\nu = 1, 2$ where

$$t \in \mathbb{C} \setminus \{0\} \qquad (q^6 + q^4 + 1)t - (q^6 + q^4 + q^2) \neq 0$$

in the first and

$$t \in \mathbb{C} \setminus \{0\} \qquad (q^6 + q^2 + 1)t - (q^4 + q^2 + 1) \neq 0$$

in the second case. The calculuses $\Gamma_\nu(t)$ and $\Gamma_{\nu'}(t')$ are equivalent if and only if $\nu = \nu'$ and $t = t'$.

Remark. The calculuses can be described explicitly in terms of their fundamental matrices $\mathcal{F}(z^i{}_j)$ $(i, j = 1, 2, 3)$ which depend on q and the extra parameter t. The rather lengthy expressions can be found in [23]. For the exceptional values $q = \pm 1, \pm\mathrm{i}$ there may be further calculuses.

Now one can calculate the commutation relations of the generators of $\mathcal{A}$ and their differentials from the commutation relations involving the Maurer–Cartan 1-forms. These calculations have also been carried out with the help of *Reduce*.

Corollary 4.3. Let $q \in \mathbb{C} \setminus \{0, \pm 1, \pm\mathrm{i}\}$. The bicovariant differential calculuses $\Gamma_1(t)$ on $GL_q(3,\mathbb{C})$ are given by

$$
\begin{aligned}
(\mathrm{d}z^i{}_j)z^i{}_j &= \left(\frac{t}{q^2} + t - 1\right) z^i{}_j\,\mathrm{d}z^i{}_j + \sigma z^i{}_j z^i{}_j \mathrm{Tr}_q\theta \\
(\mathrm{d}z^i{}_j)z^i{}_l &= \frac{t}{q} z^i{}_l\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^i{}_l + \sigma z^i{}_j z^i{}_l \mathrm{Tr}_q\theta \qquad j < l \\
(\mathrm{d}z^i{}_j)z^k{}_j &= \frac{t}{q} z^k{}_j\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^k{}_j + \sigma z^i{}_j z^k{}_j \mathrm{Tr}_q\theta \qquad i < k \\
(\mathrm{d}z^i{}_j)z^k{}_l &= t z^k{}_l\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^k{}_l + \sigma z^i{}_j z^k{}_l \mathrm{Tr}_q\theta \\
&\quad - \beta(z^i{}_j z^k{}_l - q z^i{}_l z^k{}_j)\mathrm{Tr}_q\theta \qquad i < k,\ j < l \\
(\mathrm{d}z^i{}_j)z^k{}_l &= t z^k{}_l\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^k{}_l + \sigma z^i{}_j z^k{}_l \mathrm{Tr}_q\theta - t\left(q - \frac{1}{q}\right) z^k{}_j \mathrm{d}z^i{}_l \\
&\quad + q\beta(z^i{}_l z^k{}_j - q z^i{}_j z^k{}_l)\mathrm{Tr}_q\theta \qquad i < k,\ j > l
\end{aligned}
\tag{4.8}
$$

with $t \in \mathbb{C} \setminus \{0\}$, $(q^6 + q^4 + 1)t - (q^6 + q^4 + q^2) \neq 0$. The second family of calculuses $\Gamma_2(t)$ is determined by

$$
\begin{aligned}
(\mathrm{d}z^i{}_j)z^i{}_j &= (tq^2 + t - 1) z^i{}_j\,\mathrm{d}z^i{}_j + \hat{\sigma} z^i{}_j z^i{}_j \mathrm{Tr}_q\theta \\
(\mathrm{d}z^i{}_j)z^i{}_l &= tq z^i{}_l\,\mathrm{d}z^i{}_j + (tq^2 - 1) z^i{}_j\,\mathrm{d}z^i{}_l + \hat{\sigma} z^i{}_j z^i{}_l \mathrm{Tr}_q\theta \qquad j < l \\
(\mathrm{d}z^i{}_j)z^k{}_j &= tq z^k{}_j\,\mathrm{d}z^i{}_j + (tq^2 - 1) z^i{}_j\,\mathrm{d}z^k{}_j + \hat{\sigma} z^i{}_j z^k{}_j \mathrm{Tr}_q\theta \qquad i < k \\
(\mathrm{d}z^i{}_j)z^k{}_l &= t z^k{}_l\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^k{}_l + \hat{\sigma} z^i{}_j z^k{}_l \mathrm{Tr}_q\theta \\
&\quad + t\left(q - \frac{1}{q}\right)(z^i{}_l\,\mathrm{d}z^k{}_j + z^k{}_j\,\mathrm{d}z^i{}_l) + t\left(q - \frac{1}{q}\right)^2 z^i{}_j\,\mathrm{d}z^k{}_l \\
&\quad - \frac{1}{q^2}\hat{\beta}(z^i{}_j z^k{}_l - q z^i{}_l z^k{}_j)\mathrm{Tr}_q\theta \qquad i < k,\ j < l \\
(\mathrm{d}z^i{}_j)z^k{}_l &= t z^k{}_l\,\mathrm{d}z^i{}_j + (t-1) z^i{}_j\,\mathrm{d}z^k{}_l + \hat{\sigma} z^i{}_j z^k{}_l \mathrm{Tr}_q\theta + t\left(q - \frac{1}{q}\right) z^i{}_l\,\mathrm{d}z^k{}_j \\
&\quad + \frac{1}{q}\hat{\beta}(z^i{}_l z^k{}_j - q z^i{}_j z^k{}_l)\mathrm{Tr}_q\theta \qquad i < k,\ j > l
\end{aligned}
\tag{4.9}
$$

with $t \in \mathbb{C} \setminus \{0\}$, $(q^6 + q^2 + 1)t - (q^4 + q^2 + 1) \neq 0$. We have introduced the abbreviations

$$\mathrm{Tr}_q\theta = q^4\theta^1{}_1 + q^2\theta^2{}_2 + \theta^3{}_3 \tag{4.10}$$

$$\sigma = \frac{(q^2 - t)(t - 1)}{q^4(q^2 + 1)(t - 1) - q^2 + t} \tag{4.11}$$

$$\beta = \frac{t(q^2-1)(t-1)}{q^4(q^2+1)(t-1)-q^2+t} \tag{4.12}$$

$$\hat{\sigma} = -\frac{(q^2t-1)(t-1)}{q^4(q^2t-1)+(q^2+1)(t-1)} \tag{4.13}$$

$$\hat{\beta} = -\frac{t(q^2-1)(t-1)}{q^4(q^2t-1)+(q^2+1)(t-1)}\,. \tag{4.14}$$

The missing relations can be derived in both cases by using the Leibniz rule and the relations (3.1).

Remark. For the special case $t = 1$ the formulae (4.8) and (4.9) simplify drastically. They can be written in compact form

$$\mathrm{d}Z_1\, Z_2 = R_{12}^{-1} Z_2\, \mathrm{d}Z_1\, R_{21}^{-1} \tag{4.15}$$

$$\mathrm{d}Z_1\, Z_2 = R_{21} Z_2\, \mathrm{d}Z_1\, R_{12} \tag{4.16}$$

for $\nu = 1$ and $\nu = 2$, respectively, and define bicovariant differential calculuses for arbitrary n. These relations were first found by Maltsiniotis [20] and independently by Manin [21]. They investigated differential calculuses on multi-parameter deformations of $GL(n)$ that are induced by calculuses on the corresponding quantum plane. In R-matrix form (4.15) and (4.16) appeared in [24–26] and were studied in detail in [27] (see also [28]).

The bi-invariant element $\mathrm{Tr}_q\,\theta$ plays a particular role. Acting with it on $\mathcal{A}$ by taking the commutator $[\mathrm{Tr}_q\,\theta, a]$ ($a \in \mathcal{A}$) defines a derivation from $\mathcal{A}$ into the space of 1-forms. It turns out that this derivation coincides with d up to a normalization factor.

Proposition 4.4. For all first-order differential calculuses $(\Gamma_\nu(t), \mathrm{d})$ on $GL_q(3, \mathbb{C})$ the differential d is an inner derivation:

$$\mathrm{d}a = \frac{1}{\mathcal{N}}\,[\mathrm{Tr}_q\theta, a] \tag{4.17}$$

where

$$\mathcal{N} = \begin{cases} \dfrac{1}{q^2}(q^4(q^2+1)(t-1)-q^2+t) & \text{for} \quad \nu = 1 \\ q^4(q^2t-1)+(q^2+1)(t-1) & \text{for} \quad \nu = 2\,. \end{cases} \tag{4.18}$$

4.4. *R-matrix formulation*

The commutation relations of $GL_q(3, \mathbb{C})$ can be written in the compact form (3.5) using the R-matrix (3.6). Now the question arises whether also the bimodule structure of $\Gamma_\nu(t)$ can be compactly expressed in such a way. Indeed, this can be achieved by using a convenient basis of ${}_{\mathrm{inv}}\Gamma$. It is related to a procedure proposed by Jurčo [29] to construct bicovariant differential calculuses on certain (classes of) quantum groups. The latter can be applied to the case of $GL_q(n, \mathbb{C})$ for arbitrary dimension n. The construction is based on a further result of Woronowicz [12] which we recall next.

Given a family of functionals $f = (f^I{}_J)_{I,J\in\mathcal{I}}$ and a family of algebra elements $M = (M_I{}^J)_{I,J\in\mathcal{I}}$ satisfying (2.9), (2.10), (2.12), (2.13) and the compatibility condition (2.14) one can endow the free left $\mathcal{A}$ -module Γ generated by $\{\omega^I\}_{I\in\mathcal{I}}$ with the structure of a bicovariant bimodule: one regards $\{\omega^I\}$ as left-invariant elements forming a basis of ${}_{\mathrm{inv}}\Gamma$ and defines the right multiplication by (2.8) and the right coaction by (2.11).

It is easy to see that $M = Z$ and $M = S(Z)^t$ are possible choices for M (^{t} denotes ordinary matrix transposition). The appropriate functionals are the generators

$L^{\pm} = (\ell^{\pm i}{}_j)_{1\leqslant i,j\leqslant n}$ of the algebra of regular functionals on $GL_q(n,\mathbb{C})$. They are defined by [22]

$$\begin{aligned}\langle \ell^{\pm i}{}_j, z^k{}_l\rangle &= R^{\pm}{}^{ik}_{jl}\\ \langle \ell^{\pm i}{}_j, 1\rangle &= \delta^i_j\\ \langle \ell^{\pm i}{}_j, ab\rangle &= \langle \ell^{\pm i}{}_k, a\rangle\langle \ell^{\pm k}{}_j, b\rangle\end{aligned} \tag{4.19}$$

for all $a, b \in \mathcal{A}$ where we denote the evaluation $\ell(a)$ by $\langle \ell, a\rangle$ and use the abbreviations

$$R^+ = c^+ PRP \qquad R^- = c^- R^{-1}. \tag{4.20}$$

Here P is the permutation matrix $P^{ik}{}_{jl} = \delta^i_l\delta^k_j$ and c^+, c^- are complex constants $\neq 0$. The quantum Yang–Baxter equation (3.7) assures the compatibility of (4.19) with the relations (3.5). The dual of $\mathcal{A}$ denoted by $\mathcal{A}'$ has a natural multiplication structure given by the convolution product

$$\langle f * g, a\rangle = \langle f\otimes g, \Delta(a)\rangle \qquad a\in\mathcal{A}, f, g\in\mathcal{A}'$$

and contains ε as unit element. One regards the subalgebra $\mathcal{U}$ of $\mathcal{A}'$ generated by $\ell^{\pm i}{}_j$ (and two further functionals $\ell^{\pm}$ playing a similar role as the inverse of the quantum determinant in the construction of the Hopf algebra $\mathcal{A}$). $\mathcal{U}$ can be endowed with the structure of a Hopf algebra in a natural way (cf [22, 30]). In particular, one obtains for the antipode S'

$$\langle S'(L^{\pm}), Z\rangle = \langle L^{\pm}, S(Z)\rangle = (R^{\pm})^{-1}. \tag{4.21}$$

It turns out that in the case of $M = Z$ the choice $f = S'(L^{\pm})^t$ fulfils all requirements mentioned above. The condition (2.14) is checked on the generators $a = z^i{}_j$ with the help of the basic relations (3.5). For $M = S(Z)^t$ one sets $f = L^{\pm}$. However, in these cases one is led to bicovariant bimodules of dimension n. To build up an n^2-dimensional bimodule as a candidate for a differential calculus on $GL_q(n,\mathbb{C})$ tensor products of two n-dimensional bimodules can be used. Out of the various possibilities [29] we choose

$$M^I{}_J = M^i{}_{jk}{}^l = z^i{}_k S(z^l{}_j) \qquad f_I{}^J = f_i{}^{jk}{}_l = S'(\ell^{\pm k}{}_i) * \ell^{\mp j}{}_l. \tag{4.22}$$

The commutation relations of the bimodule generators $\omega_i{}^j$ and the algebra generators $z^k{}_l$ are for the choice of upper signs in (4.22)

$$\omega_i{}^j z^k{}_l = t z^k{}_d (R^{-1})^{da}{}_{ei}\ (R^{-1})^{je}{}_{bl}\ \omega_a{}^b \qquad (t = c^-/c^+ \neq 0) \tag{4.23}$$

and in the case of lower signs

$$\omega_i{}^j z^k{}_l = t z^k{}_d R^{ad}{}_{ie}\ R^{ej}{}_{lb}\ \omega_a{}^b \qquad (t = c^+/c^- \neq 0). \tag{4.24}$$

These have the desired simple form.

To introduce a differential operator d one uses (in both cases) the bi-invariant element $\mathrm{Tr}\,\omega = \sum_i \omega_i{}^i$. $\mathrm{d}a$ is defined for all $a\in\mathcal{A}$ as

$$\mathrm{d}a = \frac{1}{q-q^{-1}}[\mathrm{Tr}\,\omega, a] \tag{4.25}$$

where d satisfies the Leibniz rule and using the bi-invariance of $\mathrm{Tr}\,\omega$ one can verify (2.2) and (2.3). Now it is possible to calculate the relation between $\omega_i{}^j$ and the Maurer–Cartan 1-forms defined in (4.3). One obtains†

$$\theta^i{}_j = U^{ik}{}_{jl}\ \omega_k{}^l \tag{4.26}$$

† Here the double index ${}^i{}_j$ determines the row and ${}^k{}_l$ the column of the matrix U.

where the complex matrix $U \in M(n^2, \mathbb{C})$ is given by

$$U^{ik}_{\ jl} = \frac{1}{q-q^{-1}}(t(R^{-1})^{ik}_{\ ab}\ (R^{-1})^{ba}_{\ lj} - \delta^i_j\delta^k_l) \tag{4.27}$$

$$U^{ik}_{\ jl} = \frac{1}{q-q^{-1}}(tR^{ki}_{\ ab}\ R^{ba}_{\ jl} - \delta^i_j\delta^k_l) \tag{4.28}$$

in the first and second case, respectively. Γ is generated by $\mathrm{d}z^i{}_j$ as a left $\mathcal{A}$ -module† if and only if U is invertible. This leads to additional restrictions on t, in the case $n = 3$ these are

$$(q^6+q^4+1)t - (q^6+q^4+q^2) \neq 0 \qquad \text{for} \quad \nu = 1$$
$$(q^6+q^2+1)t - (q^4+q^2+1) \neq 0 \qquad \text{for} \quad \nu = 2\,.$$

Using the transformation (4.26), the relations (4.23) and (4.24) lead to commutation relations of Maurer–Cartan 1-forms and algebra generators which agree with those found in section 4.3 for the differential calculuses $\Gamma_\nu(t)$.

Proposition 4.5. Let $q \in \mathbb{C} \setminus \{0, \pm 1, \pm \mathrm{i}\}$. For every bicovariant differential calculus on $GL_q(3, \mathbb{C})$ there is a basis of ${}_{\mathrm{inv}}\Gamma$ such that the commutation relations (2.8) can be expressed in terms of the R-matrix as follows. For the calculuses $\Gamma_1(t)$ this basis is given by (4.26) and (4.27) and leads to relations (4.23). In the case of $\Gamma_2(t)$ the relations (4.24) are obtained with the transformation given by (4.26) and (4.28).

Remark. The procedure outlined above has been used in several papers to construct examples of bicovariant differential calculuses on quantum groups. The calculuses $\Gamma_1(t)$ are discussed in [31, 15] for $GL_q(2, \mathbb{C})$ and $GL_q(3, \mathbb{C})$‡. In [30] the calculuses $\Gamma_2(t)$ were given for $GL_q(n, \mathbb{C})$. It is interesting that this procedure already exhausts the possible bicovariant differential calculuses in the case of $GL_q(3, \mathbb{C})$. For $GL_{p,q}(2, \mathbb{C})$ this has been shown in [32]. In that case there is only one family of calculuses.

5. Induced calculuses on $SL_q(3,\mathbb{C})$ and real forms

With the complete collection of bicovariant differential calculuses on $GL_q(3, \mathbb{C})$ at hand one can proceed to investigate the induced calculuses on quantum subgroups. Those are obtained by imposing additional relations on $\mathcal{A}$ or by introducing an involution (a $*$-structure).

5.1. $SL_q(3,\mathbb{C})$ as quantum subgroup of $GL_q(3,\mathbb{C})$

The quantum group $SL_q(3, \mathbb{C})$ is obtained from $GL_q(3, \mathbb{C})$ by adding the unimodularity condition

$$\mathcal{D} = 1\,. \tag{5.1}$$

This is consistent with the Hopf algebra structure of $GL_q(3, \mathbb{C})$. As an immediate consequence we have

$$\mathrm{d}\mathcal{D} = 0 \tag{5.2}$$

for a differential calculus over $SL_q(3, \mathbb{C})$. We determine all bicovariant differential calculuses on $SL_q(3, \mathbb{C})$ which are 'induced' by a differential calculus on $GL_q(3, \mathbb{C})$. These

† Recall the additional assumption at the beginning of this section.

‡ The statement in [15] that the additional parameter is inessential is incorrect as we have shown.

are all calculuses on $GL_q(3,\mathbb{C})$ that are consistent with the additional conditions (5.1) and (5.2). Acting with $\mathcal{F}$ on (5.1) leads to

$$t^3 q^{\mp 2} = 1 \tag{5.3}$$

with $-$ for the first and $+$ for the second family of calculuses. Calculation of $\mathrm{d}\mathcal{D}$ leads to

$$\mathrm{d}\mathcal{D} = \frac{t^3 - q^2}{q^4(q^2+1)(t-1) - q^2 + t}\mathcal{D}\,\mathrm{Tr}_q\theta \tag{5.4}$$

$$\mathrm{d}\mathcal{D} = \frac{q^2 t^3 - 1}{q^4(q^2 t - 1) + (q^2+1)(t-1)}\mathcal{D}\,\mathrm{Tr}_q\theta \tag{5.5}$$

for the first and second case, respectively. All this can be summarized as follows.

Proposition 5.1. Let $q \in \mathbb{C} \setminus \{0, \pm 1, \pm \mathrm{i}\}$. In order to obtain bicovariant differential calculuses on $SL_q(3,\mathbb{C})$ from (4.8) and (4.9) one has to set $t^3 = q^2$ and $t^3 = q^{-2}$, respectively. Hereby solutions of (5.3) with

$$(q^6+q^4+1)t - (q^6+q^4+q^2) = 0 \qquad \text{for} \quad \nu = 1$$
$$(q^6+q^2+1)t - (q^4+q^2+1) = 0 \qquad \text{for} \quad \nu = 2$$

have to be excluded. Hence, for generic q there are six bicovariant differential calculuses on $SL_q(3,\mathbb{C})$.

Remark. Though (5.1) constrains the $z^i{}_j$, their differentials remain independent with regard to the left module structure. It is impossible, for example, to express $\mathrm{d}z^9$ as $\mathrm{d}z^9 = a_I\,\mathrm{d}z^I$, $I = 1, \ldots, 8$. This means that all bicovariant differential calculuses on $SL_q(3,\mathbb{C})$ given above have nine independent 1-forms. Indeed, as was shown in [33] the dimension of the space of 1-forms on $SL_q(n,\mathbb{C})$ is fixed to be 1 or n^2 if bicovariance is assumed.

5.2. *Real forms of $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$*

To obtain real forms of the quantum group $GL_q(3,\mathbb{C})$ one has to endow the underlying Hopf algebra with a $*$-structure, i.e. a linear map $* : \mathcal{A} \longrightarrow \mathcal{A}$ with

$$\begin{aligned} (ab)^* &= b^* a^* & \Delta(a^*) &= \Delta(a)^* \\ (\lambda a)^* &= \bar{\lambda} a^* & \varepsilon(a^*) &= \overline{\varepsilon(a)} \\ (a^*)^* &= a & S(S(a)^*)^* &= a \end{aligned} \tag{5.6}$$

for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$. Usually there are different choices for such a $*$-structure. We consider two of them [22]:

(i) The quantum group $GL_q(3,\mathbb{R})$ is obtained by setting

$$Z^* = Z \qquad (\mathcal{D}^{-1})^* = \mathcal{D}^{-1}\,. \tag{5.7}$$

The action of $*$ is extended to the whole algebra $\mathcal{A}$ as an antihomomorphism. For this to be welldefined, i.e. to be compatible with the relations (3.5), one has to demand $|q| = 1$.

(ii) Analogously one introduces for $q \in \mathbb{R}$ the notion of hermitian conjugation by

$$Z^* = S(Z)^t \qquad (\mathcal{D}^{-1})^* = \mathcal{D}\,. \tag{5.8}$$

and obtains the quantum unitary group $\mathrm{U}_q(3)$.

By imposing additionally the unimodularity condition (5.1) one is led to the quantum groups $SL_q(3,\mathbb{R})$ ($|q|=1$) and $SU_q(3)$ ($q\in\mathbb{R}$), respectively.

A bicovariant differential calculus on a $*$-Hopf algebra should admit an extension of the $*$-operation to the space of 1-forms Γ in such a way that (cf [12])

$$(a\varrho)^* = \varrho^* a^* \qquad (\varrho a)^* = a^*\varrho^* \qquad (\mathrm{d}a)^* = \mathrm{d}(a^*)\,. \tag{5.9}$$

As a consequence one has the compatibility of the $*$-structure with the left and right coaction of $\mathcal{A}$ on Γ:

$$\Delta_{\mathcal{L}}(\varrho^*) = \Delta_{\mathcal{L}}(\varrho)^* \qquad \Delta_{\mathcal{R}}(\varrho^*) = \Delta_{\mathcal{R}}(\varrho)^*\,. \tag{5.10}$$

Given a $*$-structure as well as a bicovariant differential calculus on $GL_q(3,\mathbb{C})$, there is at most one $*$-structure on $\theta^i{}_j$ that fulfils all requirements (5.9). We discuss the results in the case of the two examples above.

(i) In the case of (5.7) one deduces with the help of (5.9) the formula

$$(\theta^i{}_j)^* = q^{2(n-i)} f^n{}_{jk}{}^l(S(z^i{}_n))\theta^k{}_l\,. \tag{5.11}$$

For the calculuses $\Gamma_1(t)$ this reads explicitly

$$\begin{aligned}
(\theta^1{}_1)^* &= \frac{q^6}{t^2}\theta^1{}_1 + \frac{q^2}{t^2N}(t-1)(t-q^6)\operatorname{Tr}_q\theta \\
(\theta^1{}_i)^* &= \frac{q^5}{t^2}\theta^1{}_i \qquad \text{for} \quad i=2,3 \\
(\theta^2{}_2)^* &= \frac{q^2}{t^2}\theta^2{}_2 + \frac{q^2}{t^2N}\left((t-1)(1-q^6) + t\left(q^2+\frac{1}{q^2}-2\right)\right)\theta^3{}_3 \\
&\quad + \frac{q}{t^2N}(q^2t-1)(t-q^2)\operatorname{Tr}_q\theta \\
(\theta^2{}_i)^* &= \frac{q^3}{t^2}\theta^2{}_i \qquad \text{for } i=1,3 \\
(\theta^3{}_3)^* &= \frac{1}{t^2}\theta^3{}_3 + \frac{1}{t^2N}(q^2t-1)(t-q^2)\operatorname{Tr}_q\theta \\
(\theta^3{}_i)^* &= \frac{q}{t^2}\theta^3{}_i \qquad \text{for} \quad i=1,2
\end{aligned} \tag{5.12}$$

with $N = q^4(q^2+1)(t-1) - q^2 + t$. In the case of $\Gamma_2(t)$ we have similarly

$$\begin{aligned}
(\theta^1{}_1)^* &= \frac{1}{t^2}\theta^1{}_1 + \frac{1}{q^2t^2N}(q^2t-1)(t-q^2)\operatorname{Tr}_q\theta \\
(\theta^1{}_i)^* &= \frac{1}{qt^2}\theta^1{}_i \qquad \text{for} \quad i=2,3 \\
(\theta^2{}_2)^* &= \frac{1}{q^4t^2}\theta^2{}_2 + \frac{1}{q^6t^2N}(q^4t(q^2-1) - q^4(t-q^2) + t - 1)\theta^3{}_3 \\
&\quad + \frac{1}{q^6t^2N}(q^6t-1)(t-1)\operatorname{Tr}_q\theta \\
(\theta^2{}_i)^* &= \frac{1}{q^3t^2}\theta^2{}_i \qquad \text{for} \quad i=1,3 \\
(\theta^3{}_3)^* &= \frac{1}{q^6t^2}\theta^3{}_3 + \frac{1}{q^6t^2N}(q^6t-1)(t-1)\operatorname{Tr}_q\theta \\
(\theta^3{}_i)^* &= \frac{1}{q^5t^2}\theta^3{}_i \qquad \text{for} \quad i=1,2
\end{aligned} \tag{5.13}$$

with $N = q^4(q^2t-1)+(q^2+1)(t-1)$. For $*$ to be an involution it is necessary to require $|t|=1$. If ϱ is an arbitrary element of Γ with $\varrho = a_I\theta^I$ we set $\varrho^* = (\theta^I)^*(a_I)^*$. Then we can proof that (5.9) holds indeed using the commutation relations (4.6) and the property (4.17) observing that

$$\left(\frac{1}{\mathcal{N}}\mathrm{Tr}_q\,\theta\right)^* = -\frac{1}{\mathcal{N}}\mathrm{Tr}_q\,\theta\,. \tag{5.14}$$

Proposition 5.2. Let $q \in \{w \in \mathbb{C} \,|\, |w| = 1\} \setminus \{\pm1, \pm i\}$. Then all bicovariant $*$-calculuses on $GL_q(3,\mathbb{R})$ are given by (4.8) and (4.9) with the restriction $|t| = 1$ in both cases. All six calculuses on $SL_q(3,\mathbb{C})$ found for generic q are $*$-calculuses.

(ii) For $\mathrm{U}_q(3)$ the only $*$-structure on $\Gamma_\nu(t)$ is given by

$$(\theta^i{}_j)^* = -\theta^j{}_i\,. \tag{5.15}$$

Using (4.6) one proves that $(\varrho a)^* = a^*\varrho^*$ holds if and only if t is real. Again, (5.14) holds as a consequence of (5.15) and the reality of t. Hence $(\mathrm{d}a)^* = \mathrm{d}(a^*)$.

Proposition 5.3. Let $q \in \mathbb{R}\setminus\{0,\pm1\}$. All bicovariant $*$-calculuses on $U_q(3)$ are given by (4.8) or (4.9) with $t \in \mathbb{R}$. On $SU_q(3)$ these induce two bicovariant $*$-calculuses corresponding to the real solutions of $t^3 = q^{\pm2}$.

Remark. On $SU_q(2)$ one recovers the $4D_\pm$ calculuses [12]. The uniqueness of the latter has been shown in [34]. In [35,36] examples of bicovariant differential calculuses on $SU_q(n)$ for arbitrary n are given with the help of the constructive procedure outlined in section 4.4. In [36] the n calculuses corresponding to the choice of lower signs in (4.22) and the parameter values $t^n = q^{-2}$ are discussed. The authors claim that all these calculuses are $*$-calculuses. This is not true, however, for $t \notin \mathbb{R}$.

6. The classical limit

It is interesting to investigate the behavior of the differential calculuses on $GL_q(3,\mathbb{C})$ and $SL_q(3,\mathbb{C})$ in the limit $q \to 1$. One might expect the classical calculus to emerge. However, the formulae obtained for $q \to 1$ depend on the way in which the limit is performed.

In the case of $GL_q(3,\mathbb{C})$ the additional parameter t may depend on q but need not. If t and q are regarded as independent, we obtain with

$$\begin{aligned}\lim_{q\to1}\sigma &= \lim_{q\to1}\hat\sigma = \frac{1-t}{3}\\ \lim_{q\to1}\beta &= \lim_{q\to1}\hat\beta = 0\end{aligned} \tag{6.1}$$

from equations (4.8) and (4.9) a one-parameter family of calculuses on $GL(3,\mathbb{C})$:

$$\begin{aligned}
(\mathrm{d}z^i{}_j)z^i{}_j &= (2t-1)z^i{}_j\mathrm{d}z^i{}_j + \frac{1-t}{3}z^i{}_jz^i{}_j\mathrm{Tr}_q\theta \\
(\mathrm{d}z^i{}_j)z^i{}_l &= tz^i{}_l\mathrm{d}z^i{}_j + (t-1)z^i{}_j\,\mathrm{d}z^i{}_l + \frac{1-t}{3}z^i{}_jz^i{}_l\mathrm{Tr}_q\theta && j<l \\
(\mathrm{d}z^i{}_j)z^k{}_j &= tz^k{}_j\,\mathrm{d}z^i{}_j + (t-1)z^i{}_j\,\mathrm{d}z^k{}_j + \frac{1-t}{3}z^i{}_jz^k{}_j\mathrm{Tr}_q\theta && i<k \\
(\mathrm{d}z^i{}_j)z^k{}_l &= tz^k{}_l\,\mathrm{d}z^i{}_j + (t-1)z^i{}_j\,\mathrm{d}z^k{}_l + \frac{1-t}{3}z^i{}_jz^k{}_l\mathrm{Tr}_q\theta && i<k \quad j<l \\
(\mathrm{d}z^i{}_j)z^k{}_l &= tz^k{}_l\,\mathrm{d}z^i{}_j + (t-1)z^i{}_j\,\mathrm{d}z^k{}_l + \frac{1-t}{3}z^i{}_jz^k{}_l\mathrm{Tr}_q\theta && i<k \quad j>l\,.
\end{aligned} \tag{6.2}$$

For $t \to 1$ one recovers the classical calculus where $[\mathrm{d}z^i{}_j, z^k{}_l] = 0\ \forall\, i, j, k, l$. We can obtain calculuses on $SL(3,\mathbb{C})$ from (6.2) by imposing (5.1) which fixes t to be a solution of $t^3 = 1$. Apart from the classical calculus one is led in this way to two non-classical calculuses corresponding to the two primitive third roots of unity.

In the case of $SL_q(3,\mathbb{C})$ we meet with a different situation. Since t and q are related by (5.3), t is determined for $q \to 1$ up to the fact that a cubic equation for t has three solutions in the complex plane. For $t \to \xi$ and $t \to \xi^2$ with $\xi = \mathrm{e}^{(2\pi i/3)}$ one finds the same result as by setting $t = \xi$ or $t = \xi^2$ in (6.2). Here we investigate the case $t = q^{\pm 2/3} \to 1$ in some more detail:

$$\begin{aligned} &\lim_{q\to 1} \sigma = \lim_{q\to 1} \hat{\sigma} = \tfrac{1}{6} \\ &\lim_{q\to 1} \beta = \lim_{q\to 1} \hat{\beta} = \tfrac{1}{4}\,. \end{aligned} \tag{6.3}$$

This leads us in both cases (4.8), (4.9) to the following structure:

$$\left[\mathrm{d}z^i{}_j, z^k{}_l\right] = \gamma^i{}_j{}^k{}_l\,\tau \tag{6.4}$$

with the abbreviations

$$\begin{aligned} &\tau = \tfrac{3}{2}\mathrm{Tr}\,\theta = \tfrac{3}{2}\left(\theta^1{}_1 + \theta^2{}_2 + \theta^3{}_3\right) \\ &\gamma^i{}_j{}^k{}_l = \tfrac{1}{6}\left(z^i{}_l z^k{}_j - \tfrac{1}{3} z^i{}_j z^k{}_l\right). \end{aligned} \tag{6.5}$$

Using composite indices we have

$$\left[\mathrm{d}z^I, z^J\right] = \gamma^{IJ}\,\tau \qquad \tau = \tau_J\,\mathrm{d}z^J\,. \tag{6.6}$$

The symmetric matrix γ is degenerate, i.e. $\det\gamma = 0$, and satisfies $\gamma^{IJ}\tau_J = 0$. One of the 'coordinates' z^I is redundant because of the constraint $\mathcal{D} = \mathbf{1}$. We can eliminate, for example, z^9 in a certain coordinate patch, where $z^1 z^5 - z^2 z^4 \neq 0$. If we consider in (6.6) only indices $I, J = 1, \ldots, 8$, then we obtain a non-degenerate part of γ,

$$g = (\gamma^{IJ})_{1\leqslant I,J\leqslant 8} \tag{6.7}$$

with $\det g = -(z^1z^5 - z^2z^4)^2/(3\cdot 6^8) \neq 0$. The 1-form τ is still independent of the 1-forms $\mathrm{d}z^I$, $I = 1, \ldots, 8$. In particular, $\mathrm{Tr}\,\theta$ does not vanish in the classical limit.

The matrix g^{-1} gives rise to a metric

$$B = g_{IJ}\,\mathrm{d}z^I \otimes \mathrm{d}z^J \tag{6.8}$$

on $SL(3,\mathbb{C})$ (where we set $g_{IK}g^{KJ} = \delta_I^J$) which turns out to be the Cartan–Killing metric. In order to prove this we first introduce the Maurer–Cartan 1-forms $\hat{\theta}^i{}_j$ corresponding to the ordinary differential calculus on $SL(3,\mathbb{C})$. They are given by $\hat{\theta} = Z^{-1}\mathrm{d}Z$ and obey $\mathrm{Tr}\,\hat{\theta} = 0$. In terms of the basis $\{\hat{\theta}^I | I = 1, \ldots, 8\}$ of the space of 1-forms on $SL(3,\mathbb{C})$ we have

$$B = \hat{g}_{IJ}\,\hat{\theta}^I \otimes \hat{\theta}^J \tag{6.9}$$

with the coefficient matrix

$$(\hat{g}_{IJ}) = 6\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.10}$$

On the other hand, the Cartan–Killing metric κ on $SL(3,\mathbb{C})$ can be written as [37]

$$\kappa(\tilde{X},\tilde{Y}) = 6\,\mathrm{Tr}(XY) \tag{6.11}$$

where $\tilde{X}$ and $\tilde{Y}$ are the left-invariant vector fields generated by $X, Y \in s\ell(3,\mathbb{C})$. The basis $\{X_I | I = 1,\ldots,8\}$ of $s\ell(3,\mathbb{C})$ that generates vector fields dual to $\{\hat{\theta}^I | I = 1,\ldots,8\}$ is given by

$$X_i{}^j = e_i{}^j \qquad \text{for} \quad i \neq j$$
$$X_i{}^i = e_i{}^i - e_3{}^3 \qquad \text{for} \quad i = 1,2\,.$$

The matrices $e_i{}^j$ are defined by $(e_i{}^j)^k{}_l = \delta^k_i \delta^j_l$. Using (6.11) and (6.10) one easily obtains

$$\begin{aligned}\kappa &= \kappa(\tilde{X}_i{}^j, \tilde{X}_k{}^l)\,\hat{\theta}^i{}_j \otimes \hat{\theta}^k{}_l \\ &= 6(\delta^l_i\delta^j_k + \delta^j_i\delta^l_k)\,\hat{\theta}^i{}_j \otimes \hat{\theta}^k{}_l \\ &= \hat{g}_i{}^j{}_k{}^l\,\hat{\theta}^i{}_j \otimes \hat{\theta}^k{}_l\,.\end{aligned}$$

Consequently, B equals the Cartan–Killing metric, which is bi-invariant and has signature (5,3). The bicovariant differential calculuses on $SL_q(3,\mathbb{C})$ are compatible with the 'reality conditions' $(z^I)^* = z^I$, so that we obtain the same result for $SL_q(3,\mathbb{R})$, $q \to 1$. Then γ and τ form a (generalized) 'Galilei structure' on the group manifold $SL(3,\mathbb{R})$. A corresponding result for $SL_q(2,\mathbb{R})$ was obtained in [13] (see also [38]).

7. Conclusions

The way we obtained our results is not in principle, restricted to specific values of n. However, even for $n = 3$ computations are lengthy and tedious. We proved that for $GL_q(3,\mathbb{C})$ there are only two one-parameter families of bicovariant differential calculuses which both can be obtained by Jurčo's method described in section 4.4. Out of these (for generic q) there are six calculuses that are consistent with the condition of unimodularity. In this way one is led to all nine-dimensional bicovariant differential calculuses on $SL_q(3,\mathbb{C})$. The results are in accordance with those of Schmüdgen and Schüler [33], who succeeded to classify all bicovariant differential calculuses on $SL_q(n,\mathbb{C})$ (for arbitrary $n \in \mathbb{N}$) using methods different from ours. The latter are based on the relation [12] between bicovariant differential calculuses and ad-invariant right ideals of $\mathcal{A}$.

There have been attempts to construct bicovariant differential calculuses on $SL_q(n,\mathbb{C})$ with an (n^2-1)-dimensional space of 1-forms [39, 40] that are also bicovariant. This can only be achieved if one allows a deformation of the ordinary Leibniz rule for the exterior differential. The great advantage of keeping the latter is, however, its universality and simplicity.

On the other hand following the path outlined above one arrives at an interesting deformation of the ordinary calculus on $SL(n,\mathbb{R})$ that was discussed in a more general setting in [38, 41] . There it has been pointed out that similar structures can be found in the Itô calculus of stochastic differentials. Also, relations to proper time formulations of quantum theories have been established. All this hints at a possible physical relevance of the structure (6.6). For $SL(n,\mathbb{R})$ the natural group metric enters this formula. This motivates further investigations concerning a suitable generalization to the case $q \neq 1$. It seems to be reasonable that a candidate for a quantum group metric can be obtained this way. This would be a crucial step in gaining more insight into the geometry of a quantum group and could pave the way to a formulation of Kaluza–Klein theories using quantum groups as internal spaces.

After completion of this work we received a preprint by Schmüdgen [42] in which a complete classification of bicovariant differential calculuses on $GL_q(n, \mathbb{C})$ for arbitrary n is reported. Again, the methods used there are different from ours. Our discussion of the case $n = 3$ is more detailed and clarifies the relation to work by other authors. In particular, we have presented explicit formulae for the commutation relations of the algebra generators $z^i{}_j$ and their differentials. We have discussed calculuses on real forms of $GL_q(3, \mathbb{C})$ and considered the classical limit in some detail. Of most interest hereby is the geometric structure which arises in the classical limit of a bicovariant differential calculus on $SL(n, \mathbb{C})$.

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